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ON CERTAIN DETERMINANT FORMS AND THEIR APPLICATIONS.*

(FIRST PAPER.)

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Determinants no longer occupy in analysis the position of mere symbols of tabulated results. In the higher analysis they enter largely into the operations themselves. Not only are they powerful levers in these far-reaching methods of investigation, but are as well the instruments for discovering new relations which may exist between quantities and functions when bound together by their side lines. While there may be little or nothing that is new in the results here obtained, the operations serve somewhat to illustrate in an elementary manner the sweeping comprehensiveness of determinant forms.

Consider the alternant determinant function,

$$Fx = \begin{vmatrix} fx, & 1, & x, & x^2, & \ldots, & x^n \\ fa_n, & 1, & a_n, & a_n^2, & \ldots, & a_n^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ fa_0, & 1, & a_0, & a_0^2, & \ldots, & a_0^n \end{vmatrix}. \tag{1}$$

Let the function fx and its first n derived functions be finite and continuous for all values of the argument x from a_o to a_n.

Let

$$\begin{gathered} a_0 < a_1 < a_2 < a_3 \ldots < a_n, \\ b_0 < b_1 < b_2 \ldots < b_{n-1}, \\ c_0 < c_1 \ldots < c_{n-2}, \\ \cdots\cdots \\ w_0 < w_1, \\ u \end{gathered}$$

* A paper read before the Mathematical Section of the University of Virginia Philosophical Society.

be quantities such that

$$\begin{array}{l} a_r < b_r < a_{r+1}\,, \\ b_r < c_r < b_{r+1}\,, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ w_0 < u < w_1. \end{array}$$

I. Generalization of Rolle's Theorem.

The modern form of Rolle's theorem is this: "If a continuous function fx vanish when $x = a$, and also when $x = b$, then its derived function $f'x$, if also continuous, must vanish for some value of x between a and b." The proof of it goes hand in hand with the conception of a derived function.

Lagrange's form of Rolle's theorem is this: "If fx be a continuous function for all values of x between $x = a$ and $x = b$, then

$$f u = \frac{fa - fb}{a - b}\,,$$

where u is some value of x between a and b."

It is proposed to generalize this form of the theorem, and to express $f^n u$ in terms of $fa_0, \ldots, fa_n, a_0, \ldots, a_n$.

The determinant function Fx vanishes whenever x takes any one of the $n + 1$ values $a_0, \ldots, a_n$. Its first derived function, therefore, vanishes whenever x takes any one of the n values $b_0, \ldots, b_{n-1}$. So on, until finally its nth derived function vanishes for some value u of x, such that $a_0 < u < a_n$.

The nth derived function of Fx is, as may easily be seen,

$$F^n x = M f^n x + (-1)^{n+1}\, n!\, M_n,$$

where M is the minor of fx, and M_n that of x^n in the determinant Fx.

Hence

$$f^n u = (-1)^n\, n!\, \frac{M_n}{M}\,,$$

$$= n!\,\frac{\begin{vmatrix} 1, & a_n, & \ldots, & a_n^{n-1}, & fa_n \\ . & . & . & . & . \\ 1, & a_0, & \ldots, & a_0^{n-1}, & fa_0 \end{vmatrix}}{\begin{vmatrix} 1, & a_n, & \ldots, & a_n^{n-1}, & a_n^n \\ . & . & . & . & . \\ 1, & a_0, & \ldots, & a_0^{n-1}, & a_0^n \end{vmatrix}}\,,$$

which is the desired relation.

If we represent the minors of $fa_n, \ldots, fa_0$ in the determinant M_n by $A_n, \ldots, A_0$, respectively, we have

$$f^n u = \frac{n!}{M}(A_0 fa_0 + A_1 fa_1 + \ldots + A_n fa_n), \tag{2}$$

where M and the A's contain only the a's, and are therefore independent of the form of the function.

COROLLARY I.—If $fa_0 = fa_1 = \ldots = fa_n = fa$,
then

$$Fx = M(fx \pm fa),$$

and

$$F^r x = M f^r x;$$

whence the derivatives of Fx and fx vanish together.

Since

$$Fx = 0, \quad \text{for} \quad x = a_0, \ldots, a_n,$$

we have

$$F'b_0 = 0,\ F'b_1 = 0, \ldots,\ F'b_{n-1} = 0,$$

$$\therefore\ f'b_0 = 0,\ f'b_1 = 0, \ldots,\ f'b_{n-1} = 0.$$

In like manner,

$$f''c_0 = 0,\ f''c_1 = 0,\ \ldots,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f^n u = 0.$$

The same results hold in the particular case when

$$fa_0 = 0,\ fa_1 = 0,\ \ldots,\ fa_n = 0,$$

which is the extended form of Rolle's theorem due to M. Ossian Bonnet.

COROLLARY II.—If all of the different values of fa, except $fa_r, fa_s, fa_t, \ldots$, are equal to each other and to fa, we have

$$\begin{aligned} M\frac{f^n u}{n!} &= (A_0 + A_1 + \ldots + A_n) fa - (A_r + A_s + A_t) fa \\ &\quad + A_r fa_r + A_s fa_s + A_t fa_t \\ &= A_r(fa_r - fa) + A_s(fa_s - fa) + A_t(fa_t - fa), \end{aligned}$$

since $(A_0 + A_1 + \ldots + A_n)$ is equal to the determinant formed by replacing each of the fa's by unity, and therefore vanishes.

If all are equal except one, fa_r, then, writing out the binomial difference-products of the alternants,*

$$\begin{aligned} fa_r - fa &= \frac{M}{A_r}\frac{f^n u}{n!}, \\ &= (-1)^r (a_r - a_0)(a_r - a_1)\ldots(a_r - a_n)\frac{f^n u}{n!}. \end{aligned}$$

* Muir's Determinants, p. 162, *et seq.*

Corresponding results hold when the fa values vanish as a particular case Thus, putting x for a_r, and

$$fa_0 = 0, \quad fa_1 = 0, \quad \ldots, fa_n = 0,$$

we have

$$fx = (x - a_0)(x - a_1)\ldots(x - a_n)\frac{f^n u}{n!},$$

a form also due to M. Ossian Bonnet.

When $n = 1$, we have the following geometrical illustration of Lagrange's form of Rolle's theorem:

$$Fx = \begin{vmatrix} fx, & x, & 1 \\ fa_1, & a_1, & 1 \\ fa_0, & a_0, & 1 \end{vmatrix}$$

is the double area of the triangle whose base is the chord (fa_1, a_1), (fa_0, a_0) of the curve fx, the vertex of the triangle being a running point on the curve.

This area vanishes when the vertex coincides with either extremity of the chord. So that, if the function fx is finite, continuous, and single-valued between $x = a_1$ and $x = a_0$, we have the first derivative of Fx vanishing for some value u, of x, between a_0 and a_1. Hence,

$$fa_1 - fa_0 = (a_1 - a_0) f'u.$$

The chord being regarded as of constant length, the change of Fx is therefore proportional to the altitude of the triangle. As the length of the chord converges to zero, the altitude of the triangle for $x = u$ converges to the mid-ordinate of the segment of the osculating circle on that chord.

II. Interpolation Formulæ.

If all of the $n + 1$ values of fa are known except fa_r, we have by (2)

$$fa_r = -\left[\frac{A_0}{A_r}fa_0 + \frac{A_1}{A_r}fa_1 + \ldots + \frac{A_n}{A_r}fa_n\right] + \frac{M}{A_r}\frac{f^n u}{n!}.$$

Writing out the alternants into binomial difference-products and cancelling common factors in numerator and denominator of the coefficients, we have

$$(-1)^r fa_r = \frac{(a_r - a_1)(a_r - a_2)\ldots(a_r - a_n)}{(a_0 - a_1)(a_0 - a_2)\ldots(a_0 - a_n)} fa_0 + \frac{(a_r - a_0)(a_r - a_2)\ldots(a_r - a_n)}{(a_1 - a_0)(a_1 - a_2)\ldots(a_1 - a_n)} fa_1$$

$$\ldots + \frac{(a_r - a_0)(a_r - a_1)\ldots(a_r - a_{n-1})}{(a_n - a_0)(a_n - a_1)\ldots(a_n - a_{n-1})} fa_n$$

$$+ (-1)^r (a_r - a_0)(a_r - a_1)\ldots(a_r - a_n)\frac{f^n u}{n!}.$$

If we leave off the last term, which renders the equation exact, we have Lagrange's interpolation formula, the most important one in analysis, the principle of which is to substitute for the unknown function a rational integral function of the $(n-1)$th degree which has in common with it the n values $fa_0, \ldots, fa_n$. The residual term in $f^n u$ corresponds to the remainder after n terms in Taylor's formula, and shows that if the unknown function be a rational integral function of no higher degree than $n-1$, Lagrange's formula is exact, since $f^n u = 0$. If the unknown function be of the nth degree and the coefficient of x^n be unity, then the formula above gives exact results, since $f^n u = n!$.

The above result was written down merely to identify (1) with Lagrange's formula. The interpolation formula is written out at once from the original determinant; thus, letting $M_0, M_1, \ldots, M_n$ be the minors of $1, x, \ldots, x^n$ in Fx, we have

$$fx = -\left[\frac{M_0}{M} + \frac{M_1}{M}x + \frac{M_2}{M}x^2 + \ldots + \frac{M_n}{M}x^n\right] + \frac{Fx}{M} \tag{3}$$

$$= \varphi x + \frac{Fx}{M}.$$

In this, Fx vanishing for $x = a_0, \ldots, a_n$, makes fx and the rational integral function φx, of the nth degree, have $n+1$ common values. Neglecting Fx/M, we interpolate any function fx for a given x between a_0 and a_n. Computing the coefficients in φx once for all, any number of such interpolations may be rapidly made.

Fx/M being a function which vanishes for the $n+1$ values $x = a_0, \ldots, a_n$, we may write it

$$(x-a_0)(x-a_1)\ldots(x-a_n)\,\psi x.$$

The form of ψx* depending in general on that of fx.

If fx be a rational integral function of the nth degree, so also must Fx be. But Fx vanishes $n+1$ times; it is therefore zero for all values of x, and the interpolation is exact. If fx be of the $(n+1)$th degree, so is Fx. Hence ψx cannot contain x, and is constant.

If in (2) we put the factors in the coefficients of fa_1, etc. equal, so that

$$a_0 - a_1 = a_1 - a_2 = \ldots = \Delta a = (a_0 - a_n)/n = h/n;$$

we have

$$\frac{h^n}{n^n}\frac{f^n u}{n!} = \frac{fa_0}{n!} - \frac{fa_1}{(n-1)!} + \ldots + (-1)^r\frac{fa_r}{r!(n-r)!}\ldots + (-1)^n\frac{fa_n}{n!}.$$

* It is shown below that $\psi x = (-1)^{n+1}\dfrac{f^{n+1}v}{(n+1)!}$, where $a_0 < v < a_n$.

If $h/n = 1$, then

$$f^n u = fa_0 - nfa_1 + \frac{n(n-1)}{2!} fa_2 - \ldots + (-1)^n fa_n\,,$$

a formula analogous to the fundamental interpolation formula of finite differences.

In (3) the coefficient M_p/M of x^p is the same as that of x^n, if in M_n we change a_p into a_n, and multiply by $(-1)^{n-p}$.

III. Mechanical Quadrature.

The area of the curve fx is, by (3),

$$\int_{a_0}^{a_n} fx\,dx = \int_{a_0}^{a_n} \varphi x\,dx + \int_{a_0}^{a_n} \frac{Fx}{M}\,dx\,.$$

Approximately

$$\int_{a_0}^{a_n} fx\,dx = \int_{a_0}^{a_n} \varphi x\,dx\,,$$

the error being

$$\int_{a_0}^{a_n} \frac{Fx}{M}\,dx\,.$$

The curve Fx/M cuts the axis $n+1$ times at the points $x = a_0, \ldots, a_n$, and the area is made up of positive and negative portions which tend to annul each other. It seems probable that with a proper distribution of the $n-2$ intermediate points $a_1, \ldots, a_{n-1}$, and frequently of but one or more of them, the error area may be made to vanish.

Gauss (*Werke*, Vol. III, p. 203) shows how to make this distribution independently of the form of fx, when fx is a rational integral function of a degree not higher than $2n$ (see Jacobi's proof; Boole's Finite Differences, p. 52). Gauss's quadrature formula is the integration of Lagrange's interpolation formula, in which the error area is

$$\int_{a_0}^{a_n} (-1)^r (a_r - a_0) \ldots (a_r - a_n)\,\frac{f^n u}{n!}\,da_r\,.$$

As a simple illustration of the error vanishing when fx is an irrational function, the approximate formula gives exactly the quadrant of a circle (radius $= 1$) for $a_0 = 0$, $a_2 = 1$, and a_1 having some value between $\frac{37}{60}$ and $\frac{375}{600}$, the error for these values being $+\,0.0006$ and $-\,0.00165$, respectively.

Of course, if the ordinates be equidistant, the general formula reduces to the ordinary well known forms.

The most important researches in the theory of interpolation have had reference to this formula. The investigations of Minding, Christoffel, Mehler, and Jacobi (all published in Crelle's Journal; see Boole, Finite Diff., p. 56) have had in view the general problem of determining the position of the ordinates we should choose, so that the approximation may be as close as possible. There still remains much to be done in this direction. The great importance of the computation of definite integrals is sufficient cause for close research in the attempt to reduce the error area for the general definite integral.

We may examine

$$\int_{a_0}^{a_n} \frac{Fx}{M}\,dx$$

in the following manner :—

The original determinant suggests that we may put

$$Fx = \begin{vmatrix} 1, & x\,, & x^2\,, & \ldots, & x^{n+1} \\ 1, & a_n, & a_n^2, & \ldots, & a_n^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & a_0, & a_0^2, & \ldots, & a_0^{n+1} \end{vmatrix} R = \Delta R,$$

where R is some unknown function of x. When x has some fixed definite value x_0, we have

$$R_0 = \frac{Fx_0}{\Delta_0},$$

which is independent of x.

The function

$$Fx - \Delta R_0$$

vanishes when x has any one of the $n+2$ values $a_0, \ldots, a_n, x_0$. Its $(n+1)$th derivative

$$F^{n+1}x - R_0 \frac{d^{n+1}}{dx^{n+1}} \Delta\,,$$

or

$$Mf^{n+1}x - (-1)^{n+1}(n+1)!\,MR_0\,,$$

will therefore vanish for some value of x, say v, such that v lies between a_n and a_0, the greatest and least of these values.* Hence

$$R_0 = (-1)^{n+1} \frac{f^{n+1}v}{(n+1)!}.$$

Since x_0 may be any value of x between a_0 and a_n, we may drop the suffix, and put

$$R = (-1)^{n+1} \frac{f^{n+1}v}{(n+1)!}.$$

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* It is easy to show that Prof. Sylvester's general alternant

$$\begin{vmatrix} f_1x, & f_2x, & \dots, & f_{n+1}x \\ f_1x_1, & f_2x_1, & \dots, & f_{n+1}x_1 \\ \cdot & \cdot & \cdot & \cdot \\ f_1x_n, & f_2x_n, & \dots, & f_{n+1}x_n \end{vmatrix}$$

may be treated in the same manner, obtaining the relation

$$\begin{vmatrix} f_1x, & f_2x, & \dots, & f_rx, & \dots, & f_sx, & \dots, & f_{n+2}x \\ f_1x_1, & f_2x_1, & \dots, & f_rx_1, & \dots, & f_sx_1, & \dots, & f_{n+2}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1x_n, & f_2x_n, & \dots, & f_rx_n, & \dots, & f_sx_n, & \dots, & f_{n+2}x_n \\ 0, & 0, & \dots, & \varphi^{n+1}u, & \dots, & 1, & \dots, & 0 \end{vmatrix} \quad 0,$$

in which the last constituent of each column except the rth and the sth is zero, and

$$\varphi^{n+1}u = \frac{F_u^{n+1}}{\Delta_u^{n+1}},$$

where

$$F_u^{n+1} = \begin{vmatrix} f_1^{n+1}u, & f_2^{n+1}u, & \dots, & f_s^{n+1}u, & \dots, & f_{n+2}^{n+1}u \\ f_1x_1, & f_2x_1, & \dots, & f_sx_1, & \dots, & f_{n+2}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1x_n, & f_2x_n, & \dots, & f_sx_n, & \dots, & f_{n+2}x_n \end{vmatrix},$$

in which the rth column is wanting, and

$$\Delta_u^{n+1} \begin{vmatrix} f_1^{n+1}u, & f_2^{n+1}u, & \dots, & f_r^{n+1}u, & \dots, & f_{n+2}^{n+1}u \\ f_1x_1, & f_2x_1, & \dots, & f_rx_1, & \dots, & f_{n+2}x_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1x_n, & f_2x_n, & \dots, & f_rx_n, & \dots, & f_{n+2}x_n \end{vmatrix},$$

in which the sth column is wanting.

The quantity u has some value between the greatest and least of the values $x_1, \dots, x_n$, and is to be substituted for x after the $(n+1)$th differentiation.

Therefore

$$\int_{a_0}^{a_n} \frac{Fx}{M}\,dx = \int_{a_0}^{a_n} (-1)^{n+1} \frac{\Delta}{M} \frac{f^{n+1}v}{(n+1)!},$$

the same result as before.*

Any effort to reduce this, when the form of the function is known, must be made through an inquiry into the value and form of v,† which is a function of $f(x)$, a_0, a_n, and x.

Generalization of the Expansion of fx.

We begin by deducing Taylor's formula in determinant notation and then proceed to the generalization of the expansion of the function fx.

1. *Taylor's Formula.*—It is required to find a relation, if such exists, between the two values of a function, fx and $f(x+h)$, the first n integral powers of h, and the first n derivatives of fx.

Write down in a line, $f(x+h)$ followed by the successive powers of h up to the nth inclusive. Underneath this line write a second line formed by putting $h=0$ in the first line. Underneath this second line write a third line formed by differentiating the first line with respect to h, and in the result putting $h=0$. Repeat this operation until the first line has been differentiated n times. Draw the side lines, thus forming a determinant function D. This determinant D is a function of the quantities whose inter-relation is desired.

$$D = \begin{vmatrix} f(x+h), & 1, & h, & h^2, & h^3, & \dots, & h^{n-1}, & h^n \\ fx, & 1, & 0, & 0, & 0, & \dots, & 0, & 0 \\ f'x, & 0, & 1!, & 0, & 0, & \dots, & 0, & 0 \\ f''x, & 0, & 0, & 2!, & 0, & \dots, & 0, & 0 \\ f'''x, & 0, & 0, & 0, & 3!, & \dots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}x, & 0, & 0, & 0, & 0, & \dots, & (n_1-1)!, & 0 \\ f^n x, & 0, & 0, & 0, & 0, & \dots, & 0, & n! \end{vmatrix}.$$

We observe that D and each of its derivatives down to the nth, inclusive, vanishes if h vanishes.

* See Synopsis der Hoeheren Mathematik von Johann G. Hagen, Erster Band, pp. 158, 207, or Baltzer, Theorie der Determinanten, p. 87, where the alternant Fx may be found, and notes on Gauss's quadrature.

† See a paper on the remainder after n terms in Taylor's formula, by A. W. Whitcom, American Journal of Math., Vol. III, No. 4, which indirectly bears on this subject.

Factoring out $1!, 2!, \ldots, n!$, from the 2nd, 3rd, ..., $(n+1)$th columns, respectively, we have*

$$n!!\begin{vmatrix} f(x+h), & 1, & \frac{1}{1!}h, & \frac{1}{2!}h^2, & \ldots, & \frac{1}{n!}h^n \\ fx, & 1, & 0, & 0, & \ldots, & 0 \\ f'x & 0, & 1, & 0, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x, & 0, & 0, & 0, & \ldots, & 1 \end{vmatrix} - D = 0.$$

Observing that in the determinant the general term is

$$(-1)^{2r+1}\frac{h^r}{r!}f^r x,$$

we have

$$f(x+h) = fx + hf'x + \ldots + \frac{h^n}{n!}f^n x + \frac{D}{n!!}.$$

This series is convergent if D is finite, since by increasing n sufficiently we may make $D/n!!$ as small as we choose.

The form of the determinant suggests putting

$$D = n!!\frac{h^{n+1}}{(n+1)!}R,$$

where R is some *unknown* function of x and h.

When h has some definite fixed value h_0, we have

$$D_0 = n!!\frac{h_0^{n+1}}{(n+1)!}R_0,$$

or

$$R_0 = \frac{(n+1)!\,D_0}{n!!\,h_0^{n+1}},$$

which is independent of h.

Then the function

$$\begin{vmatrix} f(x+h), & 1, & \frac{1}{1!}h, & \ldots, & \frac{1}{n!}h^n, & \frac{1}{(n+1)!}h^{n+1} \\ fx, & 1, & 0, & \ldots, & 0, & 0 \\ f'x, & 0, & 1, & \ldots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x & 0, & 0, & \ldots, & 1, & 0 \\ R_0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix}$$

* I use the symbol $n!!$ to represent the double factorial product $1!\,2!\,3!\,\ldots\,n!$

and each of its derivatives to the nth, inclusive, vanish when $h = 0$. But the function vanishes when $h = h_0$. Its first derivative vanishes therefore for some value h_1 between 0 and h_0. Its second derivative vanishes also for h_2, such that $0 < h_2 < h_1$; and so on, until finally its $(n + 1)$th derivative vanishes for some value of h, say θh, such that $0 < \theta h < h_n < h_0$.

The $(n + 1)$th derivative of the above determinant is

$$f^{n+1}(x + h) - R_0;$$

hence

$$R_0 = f^{n+1}(x + \theta h).$$

We have, therefore,

$$D_0 = n!! \frac{h_0^{n+1}}{(n + 1)!} f^{n+1}(x + \theta h).$$

Dropping the suffix, because h_0 is *any* finite value of h such that fx and its first $(n + 1)$ derivatives are finite and continuous between the limits x and $x + h$, we have, finally,

$$\begin{vmatrix} f(x + h), & 1, & h, & \dots, & \frac{1}{(n+1)!}h^{n+1} \\ fx, & 1, & 0, & \dots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x, & 0, & 0, & \dots, 1, & 0 \\ f^{n+1}(x + \theta h), & 0, & 0, & \dots, 0, & 1 \end{vmatrix} = 0. \tag{4}$$

The original determinant was filled in with zeros merely in order to simplify its expansion. It might, however, have been written in the more general form

$$\begin{vmatrix} f(x + h), & 1, & (x + h), & (x + h)^2, & \dots, & (x + h)^n \\ fx, & 1, & x, & x^2, & \dots, & x^n \\ f'x, & 0, & 1, & 2x, & \dots, & nx^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x, & 0, & 0, & 0, & \dots, & n! \end{vmatrix},$$

and the result in corresponding shape.

2. *Generalization of the Expansion of fx.*—While, in substance, the preceding method is but Maclaurin's theorem, yet it may be regarded as an extension of it, inasmuch as it affords quite an independent method of expanding any function in powers of a variable. For while, after Maclaurin's method, we have successively after differentiation put the variable equal to zero, we might have put it equal to any constant, in order to make the successive derivatives

independent of the variable. Or, indeed, we may substitute for the variable, after differentiation, any series of constants,* as in the following :—

Let φx be any function of x, finite and continuous, together with its first n derivatives, for the values $a_0, \ldots, a_n$ of the variable.

Write down φx followed by the n powers of x. Underneath write these values when $x = a_0$. Underneath write the derivative of the first line, and substitute in it a_1 for x, and so on until the nth derivative is written with a_n for x. Draw the side lines and call the determinant F.

Factor out the factorials in the diagonal as before, giving the result ; thus,

$$n!! \begin{vmatrix} \varphi x, & 1, & x, & \frac{1}{2!}x^2, & \ldots, & \frac{1}{n!}x^n \\ \varphi a_0, & 1, & a_0, & \frac{1}{2!}a_0^2, & \ldots, & \frac{1}{n!}a_0^n \\ \varphi' a_1, & 0, & 1, & a_1, & \ldots, & \frac{1}{(n-1)!}a_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^n a_n & 0, & 0, & 0, & \ldots, & 1 \end{vmatrix} = F.$$

Let

$$F = (-1)^{n+1} n!! \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \ldots, & \frac{1}{n!}x^n, & \frac{1}{(n+1)!}x^{n+1} \\ 1, & a_0, & \frac{1}{2!}a_0^2, & \ldots, & \frac{1}{n!}a_0^n, & \frac{1}{(n+1)!}a_0^{n+1} \\ 0, & 1, & a_1, & \ldots, & & \frac{1}{n!}a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \ldots, & 1, & a_n \end{vmatrix} R,$$

$$= (-1)^{n+1} n!! \, \Delta R.$$

Where R is some *unknown* function of x.

When x has some fixed finite value x_0, we have

$$R_0 = (-1)^{n+1} \frac{F_0}{n!! \, \Delta_0},$$

which is independent of x.

The function

$$F - (-1)^{n+1} n!! \, \Delta R_0,$$

or

$$\begin{vmatrix} \varphi x, & 1, & x, & \ldots, & \frac{1}{(n+1)!}x^{n+1} \\ \varphi a_0, & 1, & a_0, & \ldots, & \frac{1}{(n+1)!}a_0^{n+1} \\ \varphi' a_1, & 0, & 1, & \ldots, & \frac{1}{n!}a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^n a_n, & 0, & 0, & \ldots, & 1, \quad a_n \\ R_0, & 0, & 0, & \ldots, & 0, \quad 1 \end{vmatrix}$$

* Throughout this paper I mean by constants, in the sense as employed above, any quantities, or in general any functions, which are independent of x.

vanishes when $x = a_0$, also when $x = x_0$; therefore its first derivative vanishes for some value of x, say x_1, between a_0 and x_0. Since its first derivative vanishes for $x = x_1$, and also for $x = a_1$, its second derivative must vanish for some value of x, say x_2, between x_1 and a_1. And so on, until finally its $(n+1)$th derivative vanishes for some value of x, say u, between x_n and a_n.

$$\therefore \quad R_0 = \varphi^{n+1}u.$$

Hence

$$F_0 - (-1)^{n+1} n!! \, \Delta_0 \, \varphi^{n+1}u = 0.$$

In this equation we may drop the suffix, since x_0 is but *any* fixed finite value which x may have, within the limits of continuity. We have, then, in general,

$$F - (-1)^{n+1} n!! \, \Delta \, \varphi^{n+1}u = 0,$$

or

$$\begin{vmatrix} \varphi x, & 1, & x, & \ldots, & & \frac{1}{(n+1)!}x^{n+1} \\ \varphi a_0, & 1, & a_0, & \ldots, & & \frac{1}{(n+1)!}a_0^{\,n+1} \\ \varphi' a_1, & 0, & 1, & \ldots, & & \frac{1}{n!}a_1^{\,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^n a_n, & 0, & 0, & \ldots, & 1, & a_n \\ \varphi^{n+1}u, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \tag{5}$$

We bring $\varphi^r a_r$ to the first place in the first row by $r+1$ interchanges; so we have for the corresponding term in the expansion of the determinant

$$(-1)^{r+1} A_r \, \varphi^r a_r;$$

and when transposed it becomes

$$(-r) \, A_r \, \varphi^r a_r.$$

Whence the series

$$\varphi x = \varphi a_0 - A_1 \varphi' a_1 + A_2 \varphi'' a_2 - \ldots + (-1)^n A_n \varphi^n a_n + (-1)^{n+1} A_{n+1} \varphi^{n+1} u, \tag{6}$$

in which the coefficients $A_1, \ldots, A_{n+1}$, are the minors of $\varphi a_1, \ldots, \varphi^n a_n$ in the above determinant, and are independent of the form of the function.

If all of the values $a_1, a_2, \ldots, a_n$ become equal to any one of them, say a_0, or if the values $a_1, a_2, \ldots, a_n$ become equal to x, the series becomes Taylor's expansion; and if $a_0 = a_1 = \ldots a_n = 0$, it becomes Maclaurin's. If $a_0 = 0$, and $a_1 = a_2 = \ldots = a_n = x$, we have John Bernoulli's series.

In order to avoid the tedium of expanding the determinant coefficients $A_1, \ldots, A_n$, we observe the following:—

$$A_r = \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \ldots, & & \frac{1}{r!}x^r \\ 1, & a_0, & \frac{1}{2!}a_0^2, & \ldots, & & \frac{1}{r!}a_0^r \\ 0, & 1, & a_1, & \ldots, & & \frac{1}{(r-1)!}a_1^{r-1} \\ . & . & . & . & . & . \\ 0, & 0, & 0, & \ldots, & 1, & a^{r-1} \end{vmatrix}$$

$$= -\left\{ \begin{vmatrix} x, & \frac{1}{2!}x^2, & \ldots, & & \frac{1}{r!}\,x^r \\ 1, & a_1, & \ldots, & & \frac{1}{(r-1)!}a_1^{r-1} \\ 0, & 1, & \ldots, & & \frac{1}{(r-2)!}a_2^{r-2} \\ . & . & . & . & . \\ 0, & 0, & \ldots, & 1, & a_{r-1} \end{vmatrix} - \begin{vmatrix} a_0, & \frac{1}{2!}a_0^2, & \ldots, & & \frac{1}{r!}\,a_0^r \\ 1, & a_1, & \ldots, & & \frac{1}{(r-1)!}a_1^{r-1} \\ 0, & 1, & \ldots, & & \frac{1}{(r-2)!}a_1^{r-2} \\ . & . & . & . & . \\ 0, & 0, & \ldots, & 1 & a_{r-1} \end{vmatrix} \right\}$$

$$= -\int_{a_0}^{x} \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \ldots, & & \frac{1}{(r-1)!}x^{r-1} \\ 1, & a_1, & \frac{1}{2!}a_1^2, & \ldots, & & \frac{1}{(r-2)!}a_1^{r-2} \\ 0, & 1, & a_2, & \ldots, & & \frac{1}{(r-3)!}a_2^{r-3} \\ . & . & . & . & . & . \\ 0, & 0, & 0, & \ldots, & 1, & a_{r-1} \end{vmatrix} dx.$$

But

$$A_{r-1} = \begin{vmatrix} 1, & x, & \frac{1}{2!}x^2, & \ldots, & & \frac{1}{(r-1)!}x^{r-1} \\ 1, & a_0, & \frac{1}{2!}a_0^2, & \ldots, & & \frac{1}{(r-1)!}a_0^{r-1} \\ 0, & 1, & a_1, & \ldots, & & \frac{1}{(r-2)!}a_1^{r-2} \\ . & . & . & . & . & . \\ 0, & 0, & 0, & \ldots, & 1, & a_{r-2} \end{vmatrix}.$$

Therefore, we observe that any A may be derived from the preceding one by an operation analogous to that of integration. Integrate the variable according to the regular rules for integration, and at the same time increase all the suffixes of the arbitrary constants by one. Thus, using the symbol I to denote the operation,* we have

$$A_r = -\mathrm{I}_{a_0}^{x} A_{r-1}\,dx.$$

* The symbol I and that of the inverse operator $\mathfrak{I}$ should in addition include the change of algebraic sign.

The functions $\varphi^r a_r$, independent of x, may also be included in the operation by increasing both the suffix of the argument and the index of the function by unity. Thus

$$A_r \varphi^r a_r = -\,\mathrm{I}_{a_0}^{x} A_{r-1}\,\varphi^{r-1} a_{r-1}\,dx.$$

$$= (-1)^r\, {}^r\mathrm{I}_{a_0}^{x}\, \varphi^0 a_0\,dx.$$

We write the series, finally,

$$\varphi x = \varphi a_0 + \sum_{i=0}^{i=n} {}^i\mathrm{I}_{a_0}^{x} A_i\,\varphi^i a_i\,dx + \varphi^{n+1}u\; {}^n\mathrm{I}_{a_0}^{x} A_1\,dx, \tag{7}$$

in which the symbol ${}^n\mathrm{I}dx$ means that the above defined operation is to be repeated n times.

A_r is a homogeneous function of the $r+1$ quantities $x_1, a_0, \ldots, a_{r-1}$. In general A_r may be made to vanish by giving either a_{r-1}, or $a_{r-2}, \ldots, a_1$ any one of the one, or two, ..., $(r-2)$ values which satisfy $A_r = 0$. For example,

$$A_2 = 0, \text{ when } a_1 = \tfrac{1}{2}(x + a_0).$$

To give the series arithmetical meaning, let x have some fixed definite value $a = a_0 + h$; let $a_1, \ldots, a_n$ be intermediate values in the ascending order of magnitude between the fixed limits a_0 and a, between which limits φx and its first n derivatives are finite and continuous for all values of x.

In A_r put first

$$a_1 = a_2 = \ldots = a_{r-1} = a_0\,,$$

the inferior limit. Then A_r becomes

$$(-1)^r \frac{(a-a_0)^r}{r!}.$$

Now put

$$a_1 = a_2 = \ldots, = a_{r-1} = a,$$

the superior limit. Then A_r becomes

$$(-1)^{r+1} \frac{(a-a_0)^r}{r!}.$$

By taking r sufficiently large as we may make these two values differ as little from zero as we choose.

In general the value of any co-efficient A_r will depend upon the distribution of the intermediate values $a_1, \ldots, a_{r-1}$, and may be made as small as we

please, or be made to vanish, by letting any one of the a's have a particular value. We have, for computing the A's, the formula

$$A_r = a_{r-1} A_{r-1} - \frac{1}{2!} a^2_{r-2} A_{r-2} + \ldots + (-1)^{n+1} \frac{a_0^r}{r!} + (-1)^r \frac{x^r}{r!}. \qquad (8)$$

For example, from this we obtain

$$A_1 = -(x - a_0),$$
$$A_2 = + \tfrac{1}{2!}(x^2 - a_0^2) - a_1(x - a_0),$$
$$A_3 = - \tfrac{1}{3!}(x^3 - a_0^3) + \tfrac{1}{2!} a_2 (x^2 - a_0^2) - a_1 a_2 (x - a_0) + \tfrac{1}{2!} a_1^2 (x - a_0),$$

and so on.

To investigate the relation which must hold between the a's in order that the series may be convergent, after some fixed term, we have by the above formula

$$\frac{A_r}{A_{r-1}} = a_{r-1} - \frac{1}{2} \frac{a_{r-2}}{A_{r-1}} \left[a_{r-2} A_{r-2} - \frac{1}{3} \frac{a^3_{r-3}}{a_{r-2}} A_{r-3} + \frac{1}{3.4} \frac{a^4_{r-4}}{a_{r-2}} A_{r-4} - \ldots \right]$$

$$= a_{r-1} - \tfrac{1}{2} a_{r-2} \frac{a_{r-2} A_{r-2} - \frac{1}{3} \frac{a^3_{r-3}}{a_{r-2}} A_{r-3} + \frac{1}{3.4} \frac{a^4_{r-4}}{a_{r-2}} A_{r-4} - \ldots}{a_{r-2} A_{r-2} - \frac{1}{2!} a^2_{r-3} A_{r-3} + \frac{1}{3!} a^3_{r-4} A_{r-4} - \ldots}.$$

In the ratio of the second term of the member on the right the first terms of the numerator and denominator are identical, while each succeeding term of the numerator is less than the corresponding term of the denominator, since generally $a_r > a_{r-1}$, the ratio in question is therefore less than unity.

Letting $A_r / A_{r-1} = \rho$, we have, therefore,

$$a_{r-1} > \rho > a_{r-1} - \tfrac{1}{2} a_{r-2};$$

also, à priori,

$$a_{r-1} > \rho > \tfrac{1}{2} a_{r-1}.$$

Therefore, if a_n lies between the limits 0 and + 1, or 0 and − 1, the series is convergent.

If we arrange the series according to the powers of x, we have

$$\varphi x = C_0 - C_1 x + C_2 \frac{x_2}{2!} - \ldots + (-1)^{n+1} C_{n+1} \frac{x^{n+1}}{(n+1)!}, \qquad (9)$$

wherein the determinants $C_0, \ldots, C_{n-1}$, are independent of x. We have

$$C_0 = \begin{vmatrix} \varphi a_0, & a_0, & \frac{1}{2!}a_0^2, & \ldots, & & \frac{1}{(n+1)!}a_0^{n+1} \\ \varphi' a_1, & 1, & a_1, & \ldots, & & \frac{1}{n!}a_1^n \\ \varphi'' a_2, & 0, & 1, & \ldots, & & \frac{1}{(n-1)!}a_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^n a_n, & 0, & 0, & \ldots, & 1, & a_n \\ \varphi^{n+1} u, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix}$$

$$= \varphi a_0 - a_0 \varphi' a_1 + \varphi'' a_2 \begin{vmatrix} a_0, & \frac{1}{2!}a_0^2 \\ 1, & a_1 \end{vmatrix} - \varphi''' a_3 \begin{vmatrix} a_0, & \frac{1}{2!}a_0^2, & \frac{1}{3!}a_0^3 \\ 1, & a_1, & \frac{1}{2!}a_1^2 \\ 0, & 1, & a_2 \end{vmatrix} + \ldots + \text{etc} \ldots$$

$$= \varphi a_0 - \beta_1 \varphi' a_1 + \beta_2 \varphi'' a_2 - \ldots . (-)^{n+1} \beta_{n+1} \varphi^{n+1} u,$$

with the relation

$$\beta_r = a_{r-1} \beta_{r-1} - \frac{1}{2!} a^2_{r-2} \beta_{r-2} + \ldots (-)^{r+1} \frac{a_0^r}{r!} .$$

As before,

$$\frac{\beta_r}{\beta_{r-1}} = a_{r-1} - \tfrac{1}{2} a_{r-2} + \varepsilon, \qquad 0 > \varepsilon > 1 ;$$

and

$$a_{r-1} > \beta_r / \beta_{r-1} > \tfrac{1}{2} a_{r-1}.$$

Again, we have

$$C_1 = -\varphi' a_1 + \varphi'' a_2 \begin{vmatrix} 1, & \frac{1}{2!}a_0^2 \\ 0, & a_1 \end{vmatrix} - \varphi''' a_3 \begin{vmatrix} 1, & \frac{1}{2!}a_1^2, & \frac{1}{3!}a_1^3 \\ 0, & a_1, & \frac{1}{2!}a_1^2 \\ 0, & 1, & a_2 \end{vmatrix} + \ldots$$

$$\therefore \quad C_1 x = -\int_0^x C_0 dx ; \qquad C_2 \frac{x^2}{2!} = -\int_0^x C_1 x dx.$$

$$C_{r+1} \frac{x^{r+1}}{(r+1)!} = -\int_0^x C_0 \frac{x^r}{r!} dx = (-)^{r} {}^{r+1}\int_0^x C_0 dx.$$

Hence the series* may be written

$$\varphi x = C_0 + \sum_{i=0}^{i=n} {}^{i}\int_0^x C_0 dx + \frac{x^{n\ 1}}{(n+1)!} \varphi^{n+1} u . \tag{10}$$

* Since the only condition in general imposed on the quantities $a_1, \ldots, a_n$, is that they shall be functions independent of x, a number of series may be deduced from the general one through suppositions regarding these arbitraries. While I have only noticed the simple particular cases of Taylor's, Maclaurin's, and Bernouilli's formulæ, I have prepared for publication a paper dealing with this series alone in which other forms are noticed, the consideration of which would be beyond the design of the present paper.

In the determinant (4) let φx be a rational integral function of the $(n+1)$th degree. And let

$$\varphi a_0 = 0, \quad \varphi' a_1 = 0, \quad \ldots, \quad \varphi^n a_n = 0\,;$$

so that a_0 is a root of $\varphi x = 0$, and $a_1, \ldots, a_n$ are roots of its first n derived functions respectively.

Then

$$\varphi^{n+1} u = (n+1)!\,;$$

and we have the identity

$$\varphi x = (-1)^{n+1}(n+1)! \begin{vmatrix} 1, & x, & \ldots, & \frac{1}{(n+1)!}x^{n+1} \\ 1, & a_0, & \ldots, & \frac{1}{(n+1)!}a_0^{n+1} \\ 0, & 1, & \ldots, & \frac{1}{n!}a_1^{n} \\ . & . & . & . \\ 0, & 0, & \ldots, 1, & a_n \end{vmatrix}.$$

If, therefore, we know a root of any equation and a root of each of its derivatives, we may at once write down the function from the above.

Also, since we can always factor out at sight $x - a_0$ from the right hand member of the identity, we can write down immediately the first depressed equation of $\varphi x = 0$, which has for its roots the other n roots of $\varphi x = 0$, without actually performing division.

If $x = 0$, then

$$\varphi(0) = (-1)^{n+1}(n+1)! \begin{vmatrix} a_0, & \frac{1}{2!}a_0^2, & \ldots, & \frac{1}{(n+1)!}a_0^{n+1} \\ 1, & a_1, & \ldots, & \frac{1}{n!}a_1^{n} \\ 0, & 1, & \ldots, & \frac{1}{(n-1)!}a_2^{n-1} \\ . & . & . & . \\ 0, & 0, & \ldots, 1, & a_n \end{vmatrix}$$

is a homogeneous relation between the roots, and the member on the right is the product of all the roots of $\varphi x = 0$, with their signs changed.

The most general power series of a function in terms of its variable, which can be conceived is

$$fx = ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \ldots + zx^{\zeta},$$

where $\alpha, \beta, \gamma, \ldots$ are positive integers, and $a, b, c, \ldots$ are functions independent of x.

If the series be convergent, it may extend to an infinite number of terms; otherwise the number of terms must be finite, and instead of a series we have a formula of relation.

To investigate the above relation, or its existence when there are $n+1$ power terms; write down the rows as before and factor out coefficients and factorials. Thus, we have

$$abc\ldots z a!\beta!\gamma!\ldots\zeta!\begin{vmatrix} fx, & \frac{x^\alpha}{\alpha!}, & \frac{x^\beta}{\beta!}, & \frac{x^\gamma}{\gamma!}, & \ldots, & \frac{x^\zeta}{\zeta!} \\ f\omega_0, & \frac{\omega_0^{\ \alpha}}{\alpha!}, & \frac{\omega_0^{\ \beta}}{\beta!}, & \frac{\omega_0^{\ \gamma}}{\gamma!} & \ldots, & \frac{\omega_0^{\ \zeta}}{\zeta!} \\ f'\omega_1, & \frac{\omega_1^{\ \alpha-1}}{(\alpha-1)!}, & \frac{\omega_1^{\ \beta-1}}{(\beta-1)!}, & \frac{\omega_1^{\ \gamma-1}}{(\gamma-1)!}, & \ldots, & \frac{\omega_1^{\ \zeta-1}}{(\zeta-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n\omega_n, & \frac{\omega_n^{\ \alpha-n}}{(\alpha-n)!}, & \frac{\omega_n^{\ \beta-n}}{(\beta-n)!}, & \frac{\omega_n^{\ \gamma-n}}{(\gamma-n)!}, & \ldots, & \frac{\omega_n^{\ \zeta-n}}{(\zeta-n)!} \end{vmatrix} = D.$$

Put

$$D \equiv abc\ldots z a!\beta!\gamma!\ldots\zeta!\begin{vmatrix} \frac{x^\alpha}{\alpha!}, & \frac{x^\beta}{\beta}, & \ldots, & \frac{x^\zeta}{\zeta!}, & \frac{x^{\zeta+\eta}}{(\zeta+\eta)!} \\ \frac{\omega_0^{\ \alpha}}{\alpha!}, & \frac{\omega_0^{\ \beta}}{\beta!}, & \ldots, & \frac{\omega_0^{\ \zeta}}{\zeta!}, & \frac{\omega_0^{\ \zeta+\eta}}{(\zeta+\eta)!} \\ \frac{\omega_1^{\ \alpha-1}}{(\alpha-1)!}, & \frac{\omega_1^{\ \beta-1}}{(\beta-1)!}, & \ldots, & \frac{\omega_1^{\ \zeta-1}}{(\zeta-1)!}, & \frac{\omega_1^{\ \zeta+\eta-1}}{(\zeta+\eta-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\omega_n^{\ \alpha-n}}{(\alpha-n)!}, & \frac{\omega_n^{\ \beta-n}}{(\beta-n)!}, & \ldots, & \frac{\omega_n^{\ \zeta-n}}{(\zeta-n)!}, & \frac{\omega_n^{\ \zeta+\eta-n}}{(\zeta+\eta-n)!} \end{vmatrix} R,$$

$\equiv \Delta R.$

Where R is some unknown function of x.

When $x = x_0$, we have

$$D_0 = \Delta_0 R_0,$$

a relation of quantities independent of x.

We have, if

$$\psi = D - \Delta\frac{D_0}{\Delta_0},$$

$$\begin{array}{llll} \psi = 0, & \text{for } x = x_0 & \text{and } x = \omega_0, & \\ \psi' = 0, & \text{for } x = x_1 & \text{and } x = \omega_1, & x_1 \text{ between } x_0 \text{ and } \omega_0; \\ \psi'' = 0, & \text{for } x = x_2 & \text{and } x = \omega_2, & x_2 \text{ between } x_1 \text{ and } \omega_1; \\ \cdot & \cdot & \cdot & \cdot \\ \psi^n = 0, & \text{for } x = x_n & \text{and } x = \omega_n, & x_n \text{ between } x_{n-1} \text{ and } \omega_{n-1}; \\ \psi^{n+1} = 0, & \text{for } x = u, & & u \text{ between } x_n \text{ and } \omega_n. \end{array}$$

Letting D_u^{n+1} denote the result of differentiating $n+1$ times with respect to x the determinant D and in the result replacing x by u, we have, with like meaning for Δ_u^{n+1},

$$D_u^{n+1} = \frac{\Delta_u^{n+1}}{\Delta_0} D_0 ,$$

or

$$D_0 = \frac{D_u^{n+1}}{\Delta_u^{n+1}} \Delta_0 .$$

Drop the suffix because x_0 is any value of x and put

$$\frac{D_u^{n+1}}{\Delta_u^{n+1}} = (-1)^{n+1} Fu .$$

We have finally

$$\begin{vmatrix} fx, & \dfrac{x^\alpha}{\alpha!}, & \dfrac{x^\beta}{\beta!}, & \dfrac{x^\gamma}{\gamma!}, & \ldots, & \dfrac{x^\zeta}{\zeta!}, & \dfrac{x^{\zeta+\eta}}{(\zeta+\eta)!} \\ f\omega_0, & \dfrac{{\omega_0}^\alpha}{\alpha!}, & \dfrac{{\omega_0}^\beta}{\beta!}, & \dfrac{{\omega_0}^\gamma}{\gamma!}, & \ldots, & \dfrac{{\omega_0}^\zeta}{\zeta!}, & \dfrac{{\omega_0}^{\zeta+\eta}}{(\zeta+\eta)!} \\ f'\omega_1, & \dfrac{{\omega_1}^{\alpha-1}}{(\alpha-1)!}, & \dfrac{{\omega_1}^{\beta-1}}{(\beta-1)!}, & \dfrac{{\omega_0}^{\gamma-1}}{(\gamma-1)!}, & \ldots, & \dfrac{{\omega_1}^{\zeta-1}}{(\zeta-1)!}, & \dfrac{{\omega_1}^{\zeta+\eta-1}}{(\zeta+\eta-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n\omega_n, & \dfrac{{\omega_n}^{\alpha-n}}{(\alpha-n)!}, & \dfrac{{\omega_n}^{\beta-n}}{(\beta-n)!}, & \dfrac{{\omega_n}^{\gamma-n}}{(\gamma-n)!}, & \ldots, & \dfrac{{\omega_n}^{\zeta-n}}{(\zeta-n)!}, & \dfrac{{\omega_n}^{\zeta+\eta-n}}{(\zeta+\eta-n)!} \\ Fu, & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0.$$

This is the general expression sought, and determines the forms which the coefficients a, b, c, ... must have.

More generally still, being given any $n+3$ functions of x, such as

$$\alpha, \beta, \gamma, \ldots, \theta, \ldots, \zeta, \varphi,$$

form a determinant F as before with $n+2$ of these functions omitting the last φ. Then form a new determinant exactly as before in which however we replace the rth function θ by the function φ, which was previously omitted, and call this latter determinant Δ.

Put

$$F = \Delta R,$$

R being an unknown function of x.

If $x = x_0$,

$$F_0 = \Delta_0 R_0$$

is independent of x.

The $(n+1)$th derivative of the function

$$F - \Delta \frac{R_0}{\Delta_0},$$

vanishes for some value of x, say u, which lies in magnitude between the greatest and least of the values $a, b, c, \ldots, z$. So that

$$F_u^{n+1} = \Delta_u^{n+1} \frac{R_0}{\Delta_0}.$$

Put

$$\frac{F_u^{n+1}}{\Delta_u^{n+1}} = \Psi_u^{n+1}.$$

Then, dropping the suffix, we have

$$F = \Psi_u^{n+1} \Delta.$$

Transposing the rth column to the first in F, and to the last in Δ we have the relation*

$$\begin{vmatrix} \theta, & \alpha, & \beta, & \ldots, & \zeta, & \varphi \\ \theta_a, & \alpha_a, & \beta_a, & \ldots. & \zeta_a, & \varphi_a \\ \theta'_b, & \alpha'_b, & \beta'_b, & \ldots, & \zeta'_b, & \varphi'_b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \theta_z^n, & \alpha_z^n, & \beta_z^n, & \ldots, & \zeta_z^n, & \varphi_z^n \\ \Psi_u^{n+1}, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0,$$

if the functions $\alpha, \beta, \gamma, \ldots$ and their derivatives are finite and continuous.

The great comprehensiveness of this formula is such that it would lead us altogether beyond the limits of this paper† to do more than consider one form of a particular instance of the simplest possible case; namely, that of three functions θ, α, and β.

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* While I have written out the first row in the variable as functions of x, and the second row as functions in which x is replaced by a, a quantity which is independent of x, and then proceeded with the derivatives, we might have inserted previous to differentiation a number of function rows in which x is replaced by quantities w, y, z etc, independent of x, thus making the determinant even more markedly an *alternant-differentiate*. This determinant I venture to name a "*composite*," inasmuch as it is composed partially of the form of an alternant and partially of a form which is that of a Wronskian. With this distinction, however, in the case of the latter; that the quantities under the derived functional signs may in general be entirely independent of each other and are independent of the corresponding quantity, x, under the primative functional sign.

† I have prepared a paper upon the application of the *composite* to the expansion of an arbitrary function in terms of certain transcendental functions, forms of $\sin x$ and $\cos x$, which presents many interesting points.

Here we have, changing the notation to more familiar forms, the relation

$$\frac{\varphi x\, fy - fx\, \varphi y}{fx\, \psi y - \psi x\, fy} = \frac{\varphi' u\, fy - f' u\, \varphi y}{f' u\, \psi y - \psi' u\, fy}, \tag{i}$$

where u lies between x and y.

In particular, if the function ψ is a constant, say c, then we have

$$\frac{\varphi x\, fy - fx\, \varphi y}{fx - fy} = \frac{\varphi' u\, fy - f' u\, \varphi y}{f' u}. \tag{ii}$$

We know that

$$fx - fy = (x - y) f' v,$$

where v lies between x and y; therefore

$$\frac{\varphi x\, fy - fx\, \varphi y}{x - y} = \frac{f' v}{f' u} (\varphi' u\, fy - f' u\, \varphi y). \tag{iii}$$

The member on the left is the form of Professor Cayley's resultant.

If φ is also a constant, say, b, then (iii) reduces to the familiar form of Lagrange's formula,

$$fx - fy = (x - y)\, f' v.$$

UNIVERSITY OF VIRGINIA, *Dec.* 15, 1891.

TISSERAND ON THE PRESENT CONDITION OF THE THEORY OF THE MOON.*

While preparing his "Traité de Mécanique Céleste" Tisserand has given us many valuable notes on subjects in celestial mechanics. One of the most interesting of these is this "Note sur l'État Actuel de la Théorie de la Lune," in which he has followed pretty closely Professor Newcomb's "Researches on the Motion of the Moon," but has presented, however, some important additions.

Although when Hansen's Tables of the Moon were published it was thought that the complicated problem of the Moon's motion was definitively solved, and although these tables seemed to satisfy very exactly the observations from 1750 to 1850, the period of accurate observations which Hansen had at his disposal to examine, yet soon after 1860 it became evident that the tables were in error, and that the errors were increasing.

The question how Hansen's Tables satisfied observations before 1750 was taken up by Newcomb in his Researches on the Motion of the Moon. All the available material was collected, and carefully and ably discussed by him. He finds from the eclipses of the moon given in the Almagest of Ptolemy the following corrections to the mean longitudes of Hansen's Tables:

Epoch.	*Corr. to Tables.*
-687	$-11' \pm 4'$
-381	-27 ± 5
-189	-20 ± 3
$+134$	-16 ± 4

The eclipses observed by the Arabs give

Epoch.	*Corr. to Tables.*
850	$-3'.8 \pm 2'.4$
927	-1.6 ± 1.7
986	-4.5 ± 1.3

For the epochs after 1620 Newcomb finds the following corrections:

1625	$+50'' \pm 13''$	1775	$0'' \pm 1''$
1650	$+39 \pm 5$	1800	0 ± 1
1675	$+32 \pm 1$	1825	0 ± 1
1700	$+21 \pm 1$	1850	0 ± 1
1725	$+7 \pm 1$	1875	-8 ± 1
1750	0 ± 1		

* Note sur l'État Actuel de la Théorie de la Lune. Par M. F. Tisserand. Bulletin Astronomique, November, 1891.

In endeavoring to account for these quite inadmissible errors Tisserand considers the calculation of the solar perturbations, the calculation of the inequalities of long period, and the numerical determination of the constants. The coefficients of the inequalities arising from the action of the sun were determined by Hansen and Delaunay by quite different methods, and were shown by Newcomb to be in practical agreement.

For the two inequalities of long period arising from the action of Venus the values adopted by Hansen are

$$V_1 = + 15''.34 \sin(-g - 16g' + 18g'' + 33^\circ 36'), \text{ period of 273 years,}$$
$$V_2 = + 21.47 \sin(8g'' - 13g' + 4^\circ 44'), \qquad \text{period of 239 years ;}$$

while the values found theoretically by Delaunay are

$$V_1 = + 16''.34 \sin(-g - 16g' + 18g'' + 35^\circ 16'.5),$$
$$V_2 = + \ 0''.27 \sin(8g'' - 13g' - 41^\circ 48'),$$

where g, g', g'' are the mean anomalies of the Moon, the Earth, and Venus respectively.

Hansen's V_1 was found by theory, while V_2 was determined by him empirically so as to satisfy observations from 1750 to 1850. Also, Hansen has adopted $12''.17$ for the secular acceleration s, though the theoretical value determined by Adams and Delaunay is $6''.18$, the former value being supposed better to represent the ancient eclipses.

When Hansen's V_2 is added to the tabular corrections they assume the following values, failing to satisfy the observations from 1750 to 1850:

1625	+ 33″	1775	+ 21″
1650	+ 18	1800	+ 15
1675	+ 15	1825	+ 2
1700	+ 16	1850	— 11
1725	+ 16	1875	— 28
1750	+ 19		

Since the presence of the term V_2 furnished Hansen with an erroneous value of the mean motion, a correction δn is needed; also, corrections δs and δe should be given to the secular acceleration and the longitude of the epoch. But as an examination of the table just presented shows that there is no system of values of δn, δs, and δe that will make the corrections vanish, we have to assume that the present theory is incapable of representing the observations from 1625 to 1875, and all that can be done is to introduce an empirical term, such as

$$R = A \sin at + B \cos at,$$

and then seek to determine the quantities δn, δs, δe, A, B, and a, so as to satisfy the observations.

Newcomb has used for a a value such that the period of R is 273 years, which is the period of V_1, and the table last given furnishes eleven equations of condition for determining δe, δn, A, and B in terms of δs. Tisserand solves the equations four times, using for the period of R 391, 273, 209, and 170 years; and gives the residuals when δs is left undetermined, and when it is put equal to $-6''.0$; that is, when s has its theoretical value. The residuals corresponding to the period 170 years of R are larger than the modern observations would admit. The corrections to Hansen's Tables for 1889.0 are now calculated with the first three values of the period of R, and are found to be

Period of R.	*Corr.*	*Corr.*
391 years	$-20''.1 + 0''.17\,\delta s$	$-21''.1$
273 years	$-15\,.1 + 0\,.38$	$-17\,.4$
209 years	$-7\,.8 + 0\,.69$	$-11\,.9$

the values in the second column being those for which $\delta s = -6''.0$. The correction $-17''.4$ corresponding to the period 273 years agrees exactly with the observed correction. Then with this period, using the corresponding values of δe, δn, A, and B, are calculated corrections C and C' to Hansen's Tables for the years 1620–1888, both for $s = 12''.17$ and for the theoretical value $s = 6''.18$. Under O are given the corrections to the tables found from observations.

	C	O	C'	$O - C'$
1620	$+50''$	$+53''$	$+46''$	$+7''$
30	$+48$	$+48$	$+46$	$+2$
40	$+45$	$+43$	$+44$	-1
50	$+42$	$+39$	$+42$	-3
60	$+38$	$+36$	$+39$	-3
70	$+34$	$+33$	$+35$	-2
80	$+30$	$+30$	$+30$	0
1690	$+24$	$+26$	$+25$	$+1$
1700	$+20$	$+21$	$+20$	$+1$
10	$+15$	$+15$	$+15$	0
20	$+11$	$+9$	$+11$	-2
30	$+7$	$+5$	$+6$	-1
40	$+1$	$+2$	$+1$	$+1$
50	$+1$	0	0	0
60	0	0	-1	$+1$
70	-1	0	-2	$+2$
80	-2	0	-2	$+2$
90	-1	0	-1	$+1$

	C	*O*	*C′*	*O − C′*
1800	0″	0″	0″	0″
10	0	0	+ 1	− 1
20	+ 1	0	+ 2	− 2
30	+ 1	0	+ 2	− 2
40	+ 1	0	+ 2	− 2
50	0	0	+ 1	− 1
50	− 0.1	+ 0.7	+ 0.7	0.0
52	− 0.5	+ 1.3	+ 0.2	+ 1.1
54	− 1.0	+ 1.4	− 0.3	+ 1.7
56	− 1.4	+ 1.2	− 0.9	+ 2.1
58	− 1.9	+ 1.9	− 1.4	+ 3.3
60	− 2.3	+ 2.3	− 1.9	+ 4.2
62	− 2.9	+ 2.2	− 2.6	+ 4.8
64	− 3.5	+ 0.1	− 3.3	+ 3.4
66	− 4.2	− 2.3	− 4.1	+ 1.8
68	− 4.9	− 4.0	− 5.0	+ 1.0
70	− 5.7	− 5.4	− 5.9	+ 0.5
72	− 6.5	− 7.5	− 6.9	− 0.6
74	− 7.4	− 9.1	− 8.0	− 1.1
76	− 8.3	− 9.6	− 9.1	− 0.5
78	− 9.3	− 9.0	− 10.2	+ 1.2
80	− 10.3	− 10.3	− 11.4	+ 1.1
82	− 11.3	− 12.6	− 12.7	+ 0.1
84	− 12.3	− 14.8	− 14.0	− 0.8
86	− 13.4	− 15.4	− 15.4	0.0
1888	− 14.6	− 16.9	− 16.8	− 0.1

The computed corrections to the tables agree well with the observed, but there would seem to be another inequality having a coefficient of 2″ or 3″.

Having recourse to the eclipses of Ptolemy and those of the Arabs for the determination of δs, the most probable value is found to be $-5''.87$, or $s = 6''.3$, very nearly the theoretical value.

The most important addition made in this paper by Tisserand to the theory of the Moon is to point out that with the addition of an empirical term we can represent its motion by the theoretical value of the secular acceleration as well as by any. Thus, we can find as good places by using pure theory as by having regard to the friction of the tides, and supposing a diminution in the velocity of rotation of the Earth and an apparent acceleration of the movement of the Moon. Such a change in the Earth's velocity of rotation would be so small that it could hardly be measured, and though sustained by theory we cannot tell whether the change really takes place or whether it is counteracted in some way that we do not know.

The chronological eclipses, which would require s to have the value 12″,

are rejected by Newcomb and Tisserand, as the accounts of them are vague and untrustworthy.

The inequality of long period,

$$R = A \sin at + B \cos at,$$

which is needed to determine the mean motion with precision, and is introduced as an empirical term, remains to be accounted for theoretically. The combinations that can be made of the mean longitudes of the Moon, the Earth, and a planet, and those of their nodes and perihelions are very numerous, and there may be, besides V_1, and the small inequality arising from the action of Jupiter discovered by Neison, other inequalities having sensible coefficients. Also, the value of V_1, may not be correctly determined. But Tisserand thinks that in the end theory will triumph, and the law of gravitation will completely explain the Moon's motion.

ASAPH HALL, JR.

NOTE.

The principal formula demonstrated in the article "On certain space and surface integrals," pp. 61–63 of this volume, had already been given by J. Somoff. In the more general case of oblique co-ordinates it appears in his memoir "*Moyen d'exprimer directement en coordonnées curvilignes quelconques, orthogonales ou obliques, les paramètres différentiels du premier et du second ordres et la courbure d'une surface*," p. 14, *Memoirs St. Petersburg Academy*, 7 series, Vol. VIII. (1865), and is repeated in his Theoretical Mechanics with applications to several special cases. It is found on p. 13 of the second volume of Professor Ziwet's translation of the work into German, *Theoretische Mechanik*, Leipzig, Teubner, 1879. The information contained in this note was kindly supplied to the author by Professor Ziwet.

THOMAS S. FISKE.

JANUARY 16, 1892.

ON A NINE-POINT CONIC.

By Dr. Maxime Bôcher, Cambridge, Mass.

It does not seem to have been noticed that a few well-known facts, when properly stated, yield the following direct generalization of the famous nine-point circle theorem :—

Given a triangle ABC and a point P in its plane, a conic can be drawn through the following nine points :

(1) *The middle points of the sides of the triangle ;*

(2) *The middle points of the lines joining P to the vertices of the triangle ;*

(3) *The points where these last named lines cut the sides of the triangle.*

The conic possessing these properties is simply the locus of the centre of the conics passing through the four points A, B, C, P (cf. Salmon's Conic Sections, p. 153, Ex. 3, and p. 302, Ex. 15).

Moreover, if we notice that the middle points of the lines AB, AC, PB, PC form the vertices of a parallelogram inscribed in the above-mentioned nine-point conic, it follows, at once, that the lines BC and PA, being parallel respectively to two sides of this parallelogram, are conjugate chords of the conic. Hence,

Any side of the triangle and the line joining P to the opposite vertex form a pair of conjugate chords.

If each of these pairs of conjugate chords consists of two lines perpendicular to each other, the conic must become a circle. Therefore,

If the point P lies at the intersection of the perpendiculars dropped from the vertices of the triangle ABC upon the opposite sides, the nine-point conic will become the ordinary nine-point circle.

The nine-point conic will be an ellipse when P lies either within the triangle ABC or in one of the three infinite portions of the plane which can be reached from the interior of this triangle by crossing two of its bounding lines. When P lies in any of the three remaining portions of the plane the nine-point conic will be an hyperbola. When P lies *on* one of the sides (or extended sides) of the triangle ABC, we shall have not a true parabola, as we might at first sight expect, but a pair of parallel straight lines. When, however, P is at infinity, we shall have a true parabola (the line at infinity also separating those portions of the plane corresponding to ellipses from those corresponding to hyperbolæ). Finally, the case when the nine-point conic is an equilateral hyperbola is of some interest, as then the point P must lie on the circumference of the circle circumscribed about the triangle ABC.

January, 1892.

SOLUTIONS OF EXERCISES.

ACKNOWLEDGMENTS.

Joseph Bowden, Jr. 325; H. Y. Benedict 326; Geo. R. Dean 322; W. H. Echols 322; A. Hall 327; J. E. Hendricks 325; Artemas Martin 314, 320; F. Morley 321; J. F. McCulloch 323; J. C. Nagle 326; W. B. Richards 282, 316, 322; W. O. Whitescarver 320; Chas. Yardley 320; De Volson Wood 324.

314*

A CYLINDER, diameter $2b$, intersects a sphere, diameter $2a$, the surface of the cylinder passing through the centre of the sphere. Required the part of the volume of the sphere contained by the cylinder.

[*Artemas Martin*].

SOLUTION.

Taking the origin at the centre of the sphere its rectangular equation is

$$x^2 + y^2 + z^2 = a^2, \tag{1}$$

and that of the cylindric hole is

$$x^2 + y^2 = 2bx. \tag{2}$$

Also,

$$V = \int\int\int dx\, dy\, dz. \tag{3}$$

Let $x = r \cos \varphi$, and $y = r \sin \varphi$; whence (1) and (2) become

$$r^2 + z^2 = a^2, \tag{4}$$

$$r = 2b \cos \varphi, \tag{5}$$

and

$$V = \int\int\int r\, dr\, d\varphi\, dz. \tag{6}$$

The limits of z are $-\sqrt{(a^2 - r^2)}$ and $+\sqrt{(a^2 - r^2)}$; of r, $2b \cos \varphi$ and 0; of φ, $\frac{1}{2}\pi$ and 0.

$$V = 2\int\int r\sqrt{(a^2 - r^2)}\, dr\, d\varphi, = -\frac{2}{3}\int (a^2 - r^2)^{\frac{3}{2}}\, d\varphi$$

$$= \frac{4}{3}\int_0^{\frac{1}{2}\pi} a^3 d\varphi - \frac{4}{3}\int_0^{\frac{1}{2}\pi} (a^2 - 4b^2 \cos^2\varphi)^{\frac{3}{2}}\, d\varphi. \tag{7}$$

* The above solution is for the case $a > 2b$. The case $a < 2b$ remains to be treated. The case $a = 2b$ is old.—ED.

Let $\varphi = \frac{1}{2}\pi - \theta$; then $d\varphi = -d\theta$, $\cos\varphi = \sin\theta$, and (7) becomes

$$V = \frac{2}{3}\pi a^3 - \frac{4}{3}\int_0^{\frac{1}{2}\pi}(a^2 - 4b^2\sin^2\theta)^{\frac{3}{2}}\,d\theta$$

$$= \frac{2}{3}\pi a^3 - \frac{4}{3}a^2\int_0^{\frac{1}{2}\pi}\sqrt{(a^2 - 4b^2\sin^2\theta)}\,d\theta$$

$$+ \frac{16}{3}b^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(a^2 - 4b^2\sin^2\theta)}\,d\theta$$

$$= \frac{2}{3}\pi a^3 + \frac{4}{3}a^3\int_0^{\frac{1}{2}\pi}\sqrt{(1 - e^2\sin^2\theta)}\,d\theta$$

$$+ \frac{16}{3}ab^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(1 - e^2\sin^2\theta)}\,d\theta.$$

Putting $e = 2b/a$ we have

$$V = \frac{2}{3}a^3\pi - \frac{16}{9}a(a^2 - 2b^2)\,\mathrm{E}\left[\frac{1}{2}\pi, \frac{2b}{a}\right]$$

$$- \frac{4}{9}a(a^2 - 4b^2)\,\mathrm{F}\left[\frac{1}{2}\pi, \frac{2b}{a}\right].$$

When $b = \frac{1}{2}a$, $V = \frac{2}{3}a^3\left[\pi - \frac{4}{3}\right]$. [*Artemas Martin*].

321

A CIRCLE meets a hypocycloid of class 3 at six finite points. Show that the tangents to the hypocycloid at these six points touch a conic.

[*Frank Morley.*]

SOLUTION.

The equations of the hypocycloid in circular coordinates are

$$\left.\begin{aligned} x/c &= 2/t - t^2 \\ y/c &= 2t - 1/t^2 \end{aligned}\right\}, \qquad (1)$$

t being a complex quantity of modulus 1. (See a paper on the Epicycloid, American Journal, Vol. XIII, No. 2).

Any circle is

$$xy + \alpha x + \beta y + \gamma = 0.$$

Substituting from (1), we have a sextic to determine t, and we observe that the product of the roots is 1.

The tangent to (1) at t is

$$ux + vy + 1 = 0,$$

where

$$u = -t/c(1+t^3), \quad v = -t^2/c(1+t^3).$$

Let the line equation of a conic be

$$(A, B, C, F, G, H)(u, v, 1)^2 = 0.$$

Substituting for u, v in terms of t, we have again

$$\prod_1^6 t_r = 1,$$

Π denoting a product. This condition ensures that the six tangents touch a conic; and we saw that it holds if the points t are concyclic.

[*Frank Morley.*]

322

THE arc of a limaçon is shown in works on the Calculus to be equivalent to the arc of a certain ellipse. Show that the double point on the limaçon corresponds with Fagnani's point on the ellipse. [*W. B. Richards.*]

SOLUTION.

It is shown in works on the Calculus that

$$S = 2\int_0^{\frac{1}{2}\pi} \{(a+b)^2\cos^2\varphi + (a-b)^2\sin^2\varphi\}^{\frac{1}{2}}\, d\varphi,$$

is the quadrant of the ellipse on semi-axes $2(a+b)$ and $2(a-b)$, and also half of the whole length of the limaçon $r = a\cos\theta + b$.

It is well known that Fagnani's point divides the first arc into parts whose difference is $4b$, while the half difference between the two loops of the limaçon is also $4b$. [*W. H. Echols.*]

NOTE.—George R. Dean also points out that at Fagnani's point $\cos\varphi = \sqrt{\dfrac{a+b}{2a}} = \cos\frac{1}{2}\theta$ at the node of the limaçon.—ED.

323

FOR solution see *Analyst*, Vol. I, No. 1, pp. 8–9. I proposed the problem in the *Schoolday Visitor Magazine* for May, 1872, nearly two years before Mr. Siverly used it in the *Analyst*. I had forgotten these facts when I sent it for publication in the ANNALS. [*Artemas Martin.*]

335

PROVE the formula

$$x = \int_0^x \cos z \,.\, J_0(\sqrt{z^2 - x^2})\, dz,$$

in which J_0 denotes the ordinary Bessel's function of the zero order.

[*Maxime Bôcher.*]

336

IF Sylvester's difference-product determinant $\zeta^{\frac{1}{2}}(1,\ 2^2,\ 3^2, \ldots,\ n^2) = \Delta$, and if we represent this determinant when the rth column and pth row are deleted by $\Delta(r, p)$; evaluate the ratio $\Delta(r, p)/\Delta$ when n is infinite.

[*W. H. Echols.*]

337

REQUIRED the locus of the foot of the perpendicular from the centre of an ellipse upon the common chord of the ellipse and circle of curvature.

[*Artemas Martin.*]

338

PROVE synthetically that the eccentricity of a conic section is equal to the sine of the angle which the cutting plane makes with the base of the cone, divided by the sine of the angle which an element of the cone makes with the base.

[*H. B. Newson.*]

339

SHOW that the areas of the curves

$$y = px^2 + qx + r, \text{ and } y = mx^3 + px^2 + qx + r,$$

taken between the limits $x + h$ and $x - h$, are given by the formulæ

$$\Omega = 2yh + \tfrac{2}{3}ph^3, \text{ and } \Omega = 2yh + (\tfrac{2}{3}p + mx)h^3,$$

respectively.

[*W. H. Echols.*]

340

FIND the differential equation of plane curves of the third degree free from the constants of the general integral equation. [*Yale Prize Problem.*]

341

REQUIRED the form of a beam of uniform strength, supported at its ends, the weight of the beam being the only load. [*De Volson Wood.*]

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